Once upon a time, my friend Renee texted me "if you take any (ig 2 digit) number and flip the digits, the difference between the 2 numbers will always be a multiple of 9". And then: "i wonder why that is".

Claim. If n is a 2 digit number and m is the reverse of n, then 9 divides $n - m$.

Proof. Let n be a 2 digit number. Then n can be written as $10a + b$ for $a, b \in [0, \ldots, 9]$.

Let m be n with its digits reversed, $m = 10b + a$.

Then,

$$n - m = (10a + b) - (10b + a) = 9a - 9b = 9(a - b).$$

Since $a - b \in \mathbb{Z}$, then 9 divides $n - m$. □

Excellent. "i guess" says Renee, "what interests me is how this changes across bases".

Claim. If n is a 2 digit number in base γ, and m is the reverse of n, then $\gamma - 1$ divides $n - m$.

Proof. Let n be a 2 digit number in base γ. Then $n = \gamma a + b$ for $a, b \in [0, \ldots, A_{\gamma-1}]$.

I don't know conventional notation works here but I'm saying that I need 2 symbols, and the first one gets multiplied by the base which is γ, and in your system you get γ symbols, and they are $0, 1, 2, \ldots, A_{\gamma-1}$.

So in base $\gamma = 10$, $n = 10a + b$, and $a, b \in [0, \ldots, 9]$. In base $\gamma = 5$, we get $n = 5a + b$, and $a, b \in [0, \ldots, 4]$.

Let $m = \gamma b + a$. Now,

$$n - m = (\gamma a + b) - (\gamma b + a) = \gamma a - a + b - \gamma b = (\gamma - 1)a - (\gamma - 1)b = (\gamma - 1)(a - b).$$

And thus $\gamma - 1$ divides $n - m$. □

At this point I have a feeling it works for numbers with more digits than 2. To get a feel for how the argument might go for an arbitrary number of digits, I start trying out the argument for specific numbers of digits.

Claim. Let n be a 7-digit number in base γ. Then $n = a_0 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3 + a_4\gamma^4 + a_5\gamma^5 + a_6\gamma^6$ for $a_i \in [0, \ldots, A_{\gamma-1}]$.

Proof. Let m be the reverse of n, $a_6 + a_5\gamma + a_4\gamma^2 + a_3\gamma^3 + a_2\gamma^4 + a_1\gamma^5 + a_0\gamma^6$. Now,

$$\begin{aligned}
n - m &= (a_0 + a_1\gamma + a_2\gamma^2 + a_3\gamma^3 + a_4\gamma^4 + a_5\gamma^5 + a_6\gamma^6) \\
&\quad - (a_6 + a_5\gamma + a_4\gamma^2 + a_3\gamma^3 + a_2\gamma^4 + a_1\gamma^5 + a_0\gamma^6) \\
&= a_0(1-\gamma^6) + a_1(\gamma - \gamma^5) + a_2(\gamma^2 - \gamma^4) + a_3(\gamma^3 - \gamma^3) \\
&\quad + a_4(\gamma^4 - \gamma^2) + a_5(\gamma^5 - \gamma) + a_6(\gamma^6 - 1) \\
&= -a_0(\gamma^6 - 1) - a_1\gamma(\gamma^4 - 1) - a_2\gamma^2(\gamma^2 - 1) \\
&\quad + a_4\gamma^2(\gamma^2 - 1) + a_5\gamma(\gamma^4 - 1) + a_6(\gamma^6 - 1) \\
&= (a_6 - a_0)(\gamma^6 - 1) + \gamma(a_5 - a_1)(\gamma^4 - 1) + \gamma^2(a_4 - a_2)(\gamma^2 - 1) \\
&= (a_6 - a_0)(\gamma^3 + 1)(\gamma^3 - 1) + \gamma(a_5 - a_1)(\gamma^2 + 1)(\gamma^2 - 1) \\
&\quad + \gamma^2(a_4 - a_2)(\gamma + 1)(\gamma - 1) \\
&= (a_6 - a_0)(\gamma^3 + 1)(\gamma - 1)(\gamma^2 + \gamma + 1) + \gamma(a_5 - a_1)(\gamma^2 + 1)(\gamma + 1)(\gamma - 1) \\
&\quad + \gamma^2(a_4 - a_2)(\gamma + 1)(\gamma - 1) \\
n - m &= (\gamma - 1)[(a_6 - a_0)(\gamma^3 + 1)(\gamma^2 + \gamma + 1) + \gamma(a_5 - a_1)(\gamma^2 + 1)(\gamma + 1) + \gamma^2(a_4 - a_2)(\gamma + 1)]
\end{aligned}$$

And thus $\gamma - 1$ divides $n - m$. □

Now we can look at the most general case.

Claim: Let N be a k digit number in base γ, let M be the reverse of N. Then $\gamma - 1$ divides $N - M$.

Proof. Let N be a k digit number in base γ. Trivially, if $k = 1$, then $N = M$ and $N - M = 0$, and $\gamma - 1$ divides $N - M$. So suppose $k \geq 2$.

Then N can be written

$$N = a_0 + a_1\gamma + a_2\gamma^2 + \cdots + a_{k-2}\gamma^{k-2} + a_{k-1}\gamma^{k-1} = \sum_{i=0}^{k-1} a_i\gamma^i$$

for $a_i \in [0, \ldots, A_{\gamma-1}]$, i.e., a_i is one of γ different symbols.

The reverse of N is then

$$M = a_{k-1} + a_{k-2}\gamma + a_{k-3}\gamma^2 + \cdots + a_1\gamma^{k-2} + a_0\gamma^{k-1} = \sum_{i=0}^{k-1} a_i\gamma^{k-1-i}.$$

Now we can consider $N - M$:

$$\begin{aligned}
N - M &= (a_0 + a_1\gamma + \cdots + a_{k-1}\gamma^{k-1}) - (a_{k-1} + a_{k-2}\gamma + \cdots + a_0\gamma^{k-1}) &(1)\\
&= a_0(1 - \gamma^{k-1}) + a_1(\gamma - \gamma^{k-2}) + \cdots + a_{k-2}(\gamma^{k-2} - \gamma) + a_{k-1}(\gamma^{k-1} - 1) &(2)
\end{aligned}$$

We want to prove that $\gamma - 1$ divides the entire polynomial on the left. We'll do this by showing that $\gamma - 1$ divides all k terms in that sum.

Note that if k is odd, then the middle term in the above expression is $a_{(k-1)/2}(\gamma^{(k-1)/2} - \gamma^{(k-1)/2})$ which is of course 0. So if this term exists, we already know $\gamma - 1$ divides it.

For any other term, observe that each of the terms in the sum looks something like $a_i(\gamma^i - \gamma^{k-1-i})$ for some $i \in [0, \ldots, k-1]$. Let $i > k-1-i$. (If $i < k-1-i$, simply factor out a -1 and rewrite, since the sign of a_i will not change whether or not the term is divisible by $\gamma - 1$, and $i = k-1-i$ is precisely the term we showed cancels out above). Then we can rewrite this term as

$$\pm a_i \gamma^{k-1-i}(\gamma^{i-(k-1-i)} - 1) = \pm a_i \gamma^{k-1-i}(\gamma^{2i-k+1} - 1).$$

For ease of notation, let $x = 2i - k + 1$, and note that $x > 0$ since $i > k - 1 - i$. This means that each term in (2) can be written as

$$\pm a_i \gamma^{k-1-i}(\gamma^x - 1).$$

We'll focus on the $\gamma^x - 1$ factor and show that $\gamma - 1$ divides it for any x.

Case 1. If $x = 1$, then we're done, $\gamma - 1$ is a divisor of that term.

Case 2. If x is even, then $x = 2y$ for some y, and we have a difference of squares:

$$\gamma^x - 1 = \gamma^{2y} - 1 = (\gamma^y - 1)(\gamma^y + 1).$$

Now we can repeat this process with $\gamma^y - 1$; either y is 1 and we're done as in Case 1, y is even and we repeat this case again, or y is odd and we continue with Case 3.

Case 3. If x is odd, then we have a difference of odd powers:

$$\gamma^x - 1 = (\gamma - 1)(\gamma^{x-1} + \gamma^{x-2} + \cdots + \gamma + 1).$$

And we see that $\gamma - 1$ divides $\gamma^x - 1$ for odd x.

In all cases, eventually we get that $\gamma - 1$ divides $\gamma^x - 1$, and we can write each of our terms:

$$\pm a_i \gamma^{k-1-i}(\gamma^x - 1) = a_i P_i(\gamma - 1)$$

where P_i is some polynomial of γ with integer coefficients. This means we can write (2) as

$$\begin{aligned} N - M &= a_0 P_0(\gamma - 1) + a_1 P_1(\gamma - 1) + \cdots + a_{k-1} P_{k-1}(\gamma - 1) \\ &= (\gamma - 1)(a_0 P_0 + a_1 P_2 + \cdots + a_{k-1} P_{k-1}). \end{aligned}$$

It follows that $\gamma - 1$ divides $N - M$, for all k. $\square$